MATHEMATICS

REPRESENTATIONS OF MODULARY CONGRUENCE GROUPS

BY

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(Communicated by Professor H. D. KLOOSTERMAN at the meeting of May 29, 1965)

Let k be an algebraic number field. For any integral ideal $\mathfrak{m}$ in k, $\Gamma(\mathfrak{m})$ denotes the group of all 2×2 matrices $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ whose elements are integers in k and satisfy the congruences $\alpha \equiv \delta \equiv 1$, $\beta \equiv \gamma \equiv 0 \bmod \mathfrak{m}$. In [2] we studied—for the case that k is totally-real—representations of the factor group $M(\mathfrak{m}) = \Gamma(1)/\Gamma(\mathfrak{m})$. Theta-functions were introduced with the aid of a totally-imaginary quadratic extension k_1 of k. The behaviour of these functions under the substitutions $\tau \to \dfrac{\alpha\tau+\beta}{\gamma\tau+\delta}$ yielded a representation R of $M(\mathfrak{m})$. It was remarked that, if $\mathfrak{m} = \mathfrak{m}_1\mathfrak{m}_2$ is a decomposition of $\mathfrak{m}$ into relatively prime ideals, $M(\mathfrak{m})$ is the direct product of $M(\mathfrak{m}_1)$ and $M(\mathfrak{m}_2)$, hence the representations of $M(\mathfrak{m})$ are known as soon as they are known for those $\mathfrak{m}$ that are powers of prime ideals $\mathfrak{p}$. For the case $\mathfrak{m} = \mathfrak{p}^m$ it was found that the representation R depended modulo equivalence only on local properties of the extension k_1/k. Moreover, hardly any use was made of regularity properties of the theta-functions. This led us to the thought that it might be possible to find a $\mathfrak{p}$-adic analogue for the theta-functions. This turns out to be the case, as shall be shown in this article. The transformation formulae for these functions are extremely similar to those in [2] and yield again representations of $M(\mathfrak{p}^m)$. There is no need now to restrict ourselves to the case where k is totally-real.

The functions we are going to use are derived from the separate terms in a Gaussian sum. Let p be a prime number, k_p the field of p-adic numbers. Every element $\tau \in k_p$ can be written in the form $\tau = \sum\limits_{n=n_0}^{\infty} x_n p^n$, where the x_n are rational integers between 0 and $p-1$. We define the function $E(\tau)$ by $E(\tau) = \prod\limits_{n=n_0}^{\infty} e^{2\pi i x_n p^n} = \prod\limits_{n=n_0}^{-1} e^{2\pi i x_n p^n}$. It is known that $E(\tau)$ is a continuous character of the additive group of k_p, and that every continuous character is of the form $E(\varrho\tau)$, where ϱ is a fixed element of k_p. If ϱ is an integer, and τ is restricted to the group of units, the function $E\left(\dfrac{\varrho\tau}{p^m}\right)$ depends only mod p^m on τ. If $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma(p^m)$, $\dfrac{\alpha\tau+\beta}{\gamma\tau+\delta}$ is again a unit, congruent to τ mod p^m, hence this function is invariant under the substitutions of $\Gamma(p^m)$. Moreover, the functions $E\left(\frac{\varrho^2\tau}{p^m}\right)$ — the separate terms in the Gaussian sum $\sum_{\varrho \bmod p^m} E\left(\frac{\varrho^2\tau}{p^m}\right)$ — behave nicely under the substitution $\tau \to -1/\tau$, for

$$\sum_{\lambda \bmod p^m} E\left(\frac{-2\lambda\varrho}{p^m}\right) E\left(\frac{\lambda^2\tau}{p^m}\right) = E\left(-\frac{\varrho^2}{\tau p^m}\right) \sum_{\lambda \bmod p^m} E\left(\frac{(\lambda\tau-\varrho)^2}{\tau p^m}\right).$$

Here the coefficient of $E\left(-\frac{\varrho^2}{\tau p^m}\right)$ is a Gaussian sum, depending at most in its sign on τ. This factor seems to be comparable with the factor $\sqrt{1/\tau}$ occurring in the transformation formula for ordinary theta-functions.

As they stand, these functions do not seem suitable for our purpose, so apart from working over an extension $k_{\mathfrak{p}}$ of k_p we have made several other changes. The reader will recognise that these changes have been made in order to avoid sign trouble and difficulties with the case $p=2$, to have at our disposal a set of units that is transformed into itself by every substitution $\tau \to \frac{\alpha\tau+\beta}{\gamma\tau+\delta}$ of $\Gamma(1)$, and to obtain linear independence.

We shall use the following notations:

k is an algebraic number field, $\mathfrak{p}$ a prime ideal in k, $k_{\mathfrak{p}}$ the completion of k at $\mathfrak{p}$, containing the field of p-adic numbers k_p. $\Gamma(1)$ is the group of all 2×2 matrices of determinant 1 whose elements are integers in k, m is a non-negative rational integer and $\Gamma(\mathfrak{p}^m)$ the normal subgroup of $\Gamma(1)$ consisting of those $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ that are congruent to the unit matrix mod $\mathfrak{p}^m$.

For any finite extension F of $k_{\mathfrak{p}}$, $I(F)$ is the ring of integers in F, $U(F)$ the group of units in $I(F)$, $\pi(F)$ a generating element of the maximal ideal in $I(F)$, $H(F)$ the subset of $U(F)$ consisting of those τ for which the congruence $\tau \equiv \xi \bmod \pi(F)$ has no solution $\xi \in U(k_{\mathfrak{p}})$. N_F, S_F are norm and trace in F with respect to k_p.

E will be the function on k_p described above, while E_F is the function ES_F defined on F. This is a continuous character of the additive group of F.

Now, let K be a finite extension of $k_{\mathfrak{p}}$ with relative residue class degree $f>1$ and ramification degree e, L an unramified quadratic extension of K, Δ a generating element of the absolute different of K. N and S are the relative norm and trace of L/K. For any $\varrho \in L$, ϱ' is the conjugate of ϱ with respect to K. We write $\pi(k_{\mathfrak{p}})=\pi$, and $\mathbf{R}$ shall be a complete set of representatives in $I(L)$ for the ring $I(L)/(\pi^m)$. Notice that the assumption $f>1$ implies that $H(K)$ is not empty.

We shall repeatedly use the following

Lemma 1°. *Let $\lambda \in I(L)$. Then*

$$\sum_{\varrho \in \mathbf{R}} E_L\left(\frac{\varrho\lambda'}{\pi^m \Delta}\right) = \begin{cases} 0 \text{ if } \lambda \not\equiv 0 \mod \pi^m \\ N_K(\mathfrak{p})^{2m} \text{ if } \lambda \equiv 0 \mod \pi^m \end{cases}$$

2°. *Let $\varepsilon \in U(K)$. Then*

$$\sum_{\varrho \in \mathbf{R}} E_K\left(\frac{\varepsilon N(\varrho)}{\pi^m \Delta}\right) = (-1)^{em} N_K(\mathfrak{p})^m.$$

1°. follows from the fact that $E_L\left(\frac{\varrho\lambda'}{\pi^m \Delta}\right)$ is a character of the additive group of $I(L)/(\pi^m)$, equal to the principal character if and only if $\lambda \equiv 0 \mod \pi^m$ (notice that, since L/K is unramified, (Δ) is also the absolute different of L).

The proof of 2° proceeds along the familiar lines of the calculation of the square of a Gaussian sum, after writing $\pi = \varepsilon_1 \pi(K)^e$ with some $\varepsilon_1 \in U(K)$. It can be copied practically litterally from [2], p. 8 ff.

Now for every $\varrho \in \mathbf{R}$ we define the function

$$\theta_\varrho(\tau, z) = E_K\left(\frac{N(\varrho)\tau + S(\varrho z')}{\pi^m \Delta}\right),$$

where τ runs through $H(K)$ and z through $I(L)$. Evidently these functions depend modulo π^m on ϱ, τ and z.

The functions $\theta_\varrho(\tau, z)$ are linearly independent over the complex numbers. For suppose we have a relation

$$\sum_{\varrho \in \mathbf{R}} a_\varrho \theta_\varrho(\tau, z) = 0.$$

For any $\lambda \in \mathbf{R}$ we multiply this relation with $E_L\left(-\frac{\lambda z'}{\pi^m \Delta}\right)$. Summing over z, we find

$$0 = \sum_{z \in \mathbf{R}} \sum_{\varrho \in \mathbf{R}} a\ \theta_\varrho(\tau, z)\, E_L\left(-\frac{\lambda z'}{\pi^m \Delta}\right) = \sum_{\varrho \in \mathbf{R}} a_\varrho E_K\left(\frac{N(\varrho)\tau}{\pi^m \Delta}\right) \sum_{z \in \mathbf{R}} E_L\left(\frac{(\varrho - \lambda)z'}{\pi^m \Delta}\right) =$$

$$= N_K(\mathfrak{p})^{2m} a_\lambda E_K\left(\frac{N(\lambda)\tau}{\pi^m \Delta}\right)$$

by the lemma, 1°. Hence $a_\lambda = 0$.

We want to study the behaviour of these functions under the substitutions of $\Gamma(1)$. We start with the remark that, if $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma(1)$, $\tau \in H(K)$, $z \in I(L)$, we have $\alpha\tau + \beta \in H(K)$, $\gamma\tau + \delta \in H(K)$, $\frac{\alpha\tau + \beta}{\gamma\tau + \delta} \in H(K)$, $\frac{z}{\gamma\tau + \delta} \in I(L)$, since $\alpha\delta - \beta\gamma = 1$ implies $(\alpha, \beta) = 1$, $(\gamma, \delta) = 1$. Evidently we have

$$\theta_\varrho(\tau + \beta, z) = E_K\left(\frac{N(\varrho)\beta}{\pi^m \Delta}\right) \theta_\varrho(\tau, z). \tag{1}$$

Next we have

$$\sum_{\lambda\in\mathbf{R}} E_L\left(-\frac{\lambda\varrho'}{\pi^m\Delta}\right)\theta_\lambda(\tau,z) = \sum_\lambda E_K\left(\frac{N(\lambda)\tau+S(\lambda(z-\varrho)')}{\pi^m\Delta}\right) =$$

$$= \sum_\lambda E_K\left(\frac{N(\lambda\tau)+S(\lambda\tau(z-\varrho)')}{\tau\pi^m\Delta}\right).$$

Replacing $\lambda\tau+z-\varrho$ by λ, this can be written as

$$E_K\left(-\frac{N(z-\varrho)}{\tau\pi^m\Delta}\right)\sum_\lambda E_K\left(\frac{N(\lambda)}{\tau\pi^m\Delta}\right) = E_K\left(-\frac{N(z)}{\tau\pi^m\Delta}\right)(-1)^{em}N_K(\mathfrak{p})^m\,\theta_\varrho\left(-\frac{1}{\tau},\frac{z}{\tau}\right)$$

by the lemma, 2°, or

$$\theta_\varrho\left(-\frac{1}{\tau},\frac{z}{\tau}\right) = E_K\left(\frac{N(z)}{\tau\pi^m\Delta}\right)(-1)^{em}N_K(\mathfrak{p})^{-m}\sum_{\lambda\in\mathbf{R}} E_L\left(-\frac{\lambda\varrho'}{\pi^m\Delta}\right)\theta_\lambda(\tau,z). \tag{2}$$

Now, let $A = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ be an arbitrary element of $\Gamma(1)$. At least one of the elements α, γ is a unit. If γ is a unit, we have

$$\theta_\varrho\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta},\frac{z}{\gamma\tau+\delta}\right) = \theta_\varrho\left(\frac{\alpha}{\gamma}-\frac{1}{\gamma(\gamma\tau+\delta)},\frac{z}{\gamma\tau+\delta}\right) =$$

$$= E_K\left(\frac{N(\varrho)\alpha}{\gamma\pi^m\Delta}\right)\theta_\varrho\left(-\frac{1}{\gamma(\gamma\tau+\delta)},\frac{z\gamma}{\gamma(\gamma\tau+\delta)}\right)$$

by (1). Applying (2) on the right-hand side, we find

$$W\cdot E_K\left(\frac{N(\varrho)\alpha}{\gamma\pi^m\Delta}\right)\sum_{\lambda\in\mathbf{R}} E_L\left(\frac{-\lambda\varrho'}{\pi^m\Delta}\right)\theta_\lambda(\gamma(\gamma\tau+\delta),z\gamma),$$

where

$$W = E_K\left(\frac{N(z)\gamma}{(\gamma\tau+\delta)\pi^m\Delta}\right)(-1)^{em}N_K(\mathfrak{p})^{-m}.$$

Again by (1), this can be written as

$$W\cdot E_K\left(\frac{N(\varrho)\alpha}{\gamma\pi^m\Delta}\right)\sum_\lambda E_K\left(\frac{N(\lambda)\gamma\delta-S(\lambda\varrho')}{\pi^m\Delta}\right)\theta_{\lambda\gamma}(\tau,z).$$

Finally, replacing $\lambda\gamma$ by λ, we find

$$\theta_\varrho\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta},\frac{z}{\gamma\tau+\delta}\right) = W\sum_{\lambda\in\mathbf{R}} E_K\left(\frac{N(\varrho)\alpha-S(\lambda\varrho')+N(\lambda)\delta}{\gamma\pi^m\Delta}\right)\theta_\lambda(\tau,z). \tag{3}$$

If γ is not a unit, α is one, and by (2) we have

$$\theta_\varrho\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta},\frac{z}{\gamma\tau+\delta}\right)=E_K\left(\frac{-N(z)}{(\alpha\tau+\beta)(\gamma\tau+\delta)\pi^m\Delta}\right)(-1)^{em}N_K(\mathfrak{p})^{-m}\cdot$$

$$\cdot\sum_{\mu\in\mathbf{R}}E_L\left(\frac{-\mu\varrho'}{\pi^m\Delta}\right)\theta_\mu\left(\frac{\gamma\tau+\delta}{-\alpha\tau-\beta},\frac{z}{-\alpha\tau-\beta}\right).$$

Applying (3) on the right-hand side, we obtain

$$(4)\quad\begin{cases}\theta_\varrho\left(\dfrac{\alpha\tau+\beta}{\gamma\tau+\delta},\dfrac{z}{\gamma\tau+\delta}\right)=E_K\left(\dfrac{N(z)\gamma}{(\gamma\tau+\delta)\pi^m\Delta}\right)N_K(\mathfrak{p})^{-2m}\cdot\\[2ex]\qquad\cdot\displaystyle\sum_{\lambda,\mu\in\mathbf{R}}E_K\left(\frac{-N(\mu)\gamma+S((\lambda-\alpha\varrho)\mu')+N(\lambda)\beta}{\alpha\pi^m\Delta}\right)\theta_\lambda(\tau,z).\end{cases}$$

We write $(\pi^m,\gamma)=\pi^a$, and define

$$(5)\qquad\mathfrak{d}_{\sigma,\tau}=\begin{cases}0\text{ if }\sigma\not\equiv\tau\bmod\pi^a\\1\text{ if }\sigma\equiv\tau\bmod\pi^a\end{cases}$$

for any pair $\sigma,\tau\in I(L)$. Set

$$V=\sum_{\mu\in\mathbf{R}}E_K\left(\frac{-N(\mu)\gamma+S((\lambda-\alpha\varrho)\mu')}{\alpha\pi^m\Delta}\right).$$

If $\gamma\equiv0\bmod\pi^m$, we evidently have

$$(6)\qquad V=\sum_\mu E_L\left(\frac{(\lambda-\alpha\varrho)\mu'}{\alpha\pi^m\Delta}\right)=\mathfrak{d}_{\alpha\varrho,\lambda}N_K(\mathfrak{p})^{2m}.$$

If $\gamma\not\equiv0\bmod\pi^m$ we write $\gamma=\gamma_0\pi^a$, $\mu=\mu_0+\mu_1\pi^{m-a}$, and let μ_0 and μ_1 run through complete residue systems mod π^{m-a} and π^a respectively. Then we find

$$V=\sum_{\mu_0}E_K\left(\frac{-N(\mu_0)\gamma_0\pi^a+S((\lambda-\alpha\varrho)\mu_0')}{\alpha\pi^m\Delta}\right)\sum_{\mu_1}E_L\left(\frac{(\lambda-\alpha\varrho)\mu_1'}{\pi^a\Delta}\right)=$$

$$=\mathfrak{d}_{\alpha\varrho,\lambda}N_K(\mathfrak{p})^{2a}\sum_{\mu_0}E_K\left(\frac{-N(\mu_0)\gamma_0{}^2\pi^a+S((\lambda-\alpha\varrho)\mu_0')\gamma_0}{\gamma_0\alpha\pi^m\Delta}\right).$$

Writing $\lambda-\alpha\varrho=\sigma\pi^a$, and replacing $\mu_0\gamma_0$ by μ_0:

$$V=\mathfrak{d}_{\alpha\varrho,\lambda}N(\mathfrak{p})^{2a}\sum_{\mu_0}E_K\left(\frac{-N(\mu_0)+S(\sigma\mu_0')}{\gamma_0\alpha\pi^{m-a}\Delta}\right)=$$

$$=\mathfrak{d}_{\alpha\varrho,\lambda}N(\mathfrak{p})^{2a}E_K\left(\frac{N(\sigma)}{\gamma_0\alpha\pi^{m-a}\Delta}\right)\sum_{\mu_0}E_K\left(\frac{-N(\mu_0-\sigma)}{\gamma_0\alpha\pi^{m-a}\Delta}\right).$$

The sum over μ_0 is known by the lemma, 2°. We find, after replacing γ_0 and σ by $\gamma\pi^{-a}$ and $(\lambda-\alpha\varrho)\pi^{-a}$:

$$V = \mathfrak{d}_{\alpha\varrho,\lambda}(-1)^{e(m-a)} N_K(\mathfrak{p})^{m+a} E_K\left(\frac{N(\lambda-\alpha\varrho)}{\gamma\alpha\pi^m\Delta}\right). \tag{7}$$

Substituting (6) or (7) in (4) we find:

Theorem 1: *For any* $A = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \Gamma(1)$, *we have*

$$\theta_\varrho\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}, \frac{z}{\gamma\tau+\delta}\right) = E_K\left(\frac{N(z)\gamma}{(\gamma\tau+\delta)\pi^m\Delta}\right)\sum_{\lambda\in R} U_{\varrho,\lambda}\theta_\lambda(\tau, z)$$

where

$$U_{\varrho,\lambda} = \begin{cases} \mathfrak{d}_{\alpha\varrho,\lambda} E_K\left(\dfrac{N(\lambda)\beta}{\alpha\pi^m\Delta}\right) & \text{if } \gamma \equiv 0 \bmod \pi^m \\ \mathfrak{d}_{\alpha\varrho,\lambda}(-1)^{e(m-a)} N_K(\mathfrak{p})^{-m+a} E_K\left(\dfrac{N(\varrho)\alpha - S(\varrho\lambda')+N(\lambda)\delta}{\gamma\pi^m\Delta}\right) & \\ & \text{if } \gamma \not\equiv 0 \bmod \pi^m, \end{cases}$$

a is defined by $\pi^a = (\pi^m, \gamma)$, and $\mathfrak{d}_{\alpha\varrho,\lambda}$ by (5).

Now, if we define an action of A on functions $f(\tau, z)$ by

$$f(\tau, z)^A = E_K\left(\frac{-N(z)\gamma}{(\gamma\tau+\delta)\pi^m\Delta}\right) f\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}, \frac{z}{\gamma\tau+\delta}\right) \tag{8}$$

we verify easily, that for $A_1, A_2 \in \Gamma(1)$

$$(f(\tau, z)^{A_1})^{A_2} = f(\tau, z)^{A_1A_2}.$$

Since moreover the functions $\theta_\varrho(\tau, z)$ are mapped onto themselves by any $A \in \Gamma(\mathfrak{p}^m)$, it follows that $\Gamma(1)/\Gamma(\mathfrak{p}^m)$ acts as a group of linear transformations on the space spanned by the functions $\theta_\varrho(\tau, z)$, and since these were linearly independent, we have

Theorem 2: *The map* $A \to (U_{\varrho,\lambda})$ *is a matrix representation of*

$$\Gamma(1)/\Gamma(\mathfrak{p}^m).$$

Remark that the introduction of the variable z not only has served to obtain linear independence, but enables us also to distinguish between the matrices A and $-A$, since these give different values for $\dfrac{z}{\gamma\tau+\delta}$.

It is an easy exercise to calculate the trace of the matrix $(U_{\varrho,\lambda})$, which turns out to be independent of the special choice of π and Δ. Hence the above representation is modulo equivalence uniquely determined by the choice of the field K. Since it also depends on the ranges $H(K)$ and $I(L)$

of τ and z, we shall denote it by

$$R_m(K, H(K), I(L)).$$

In what follows we shall repeatedly abuse language by not distinguishing between a representation and the representation space connected with it, or between equivalent representations.

There are several ways of finding invariant subspaces of $R_m(K, H(K), I(L))$. In the first place we remark that

$$U_{\varrho, \lambda} = U_{\varepsilon\varrho, \varepsilon\lambda}$$

for every $\varepsilon \in U(L)$ with $N(\varepsilon) \equiv 1 \bmod \pi^m$. For a way of exploiting this fact, we refer the reader to [1] and [2]. Something similar can be done, if we use the fact that

$$U_{\varrho, \lambda} = U_{\varrho', \lambda'},$$

implying that the functions $\theta_\varrho(\tau, z) + \theta_{\varrho'}(\tau, z)$ span an invariant subspace, as do the functions $\theta_\varrho(\tau, z) - \theta_{\varrho'}(\tau, z)$.

We will describe a general principle by which invariant subspaces can be found. Suppose we have subsets H_0 and I_0 of $H(K)$ and $I(L)$ respectively, such that $\tau \in H_0$, $z \in I_0$ imply $\dfrac{\alpha\tau+\beta}{\gamma\tau+\delta} \in H_0$, $\dfrac{z}{\gamma\tau+\delta} \in I_0$ for every $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma(1)$. Let $R_m(K, H_0, I_0)$ be the space, spanned by the restrictions $\theta_\varrho(\tau_0, z_0)$ to H_0, I_0 of the functions $\theta_\varrho(\tau, z)$. The map

$$\varphi: \theta_\varrho(\tau, z) \to \theta_\varrho(\tau_0, z_0)$$

induces a homomorphism of $R_m(K, H(K), I(L))$ onto $R_m(K, H_0, I_0)$. Since φ commutes with all the maps A defined in (8), the kernel of φ is an invariant subspace, and its complement, equivalent to $R_m(K, H_0, I_0)$, is also a representation, in other words: $R_m(K, H_0, I_0) \subset R_m(K, H(K), I(L))$. By means of this principle we shall show

Theorem 3: 1°. *If* $m \geqslant 2$, $R_{m-2}(K, H(K), I(L)) \subset R_m(K, H(K), I(L))$.

2°. *If K_0 is an extension of $k_\mathfrak{p}$, not totally ramified over $k_\mathfrak{p}$ and properly contained in K, L_0 an unramified quadratic extension of K_0, we have*

$$R_{m-h}(K_0, H(K_0), I(L_0)) \subset R_m(K, H(K), I(L)) \text{ for } h = 0, 1, \ldots, m.$$

To prove 1° we take $H_0 = H(K)$, $I_0 = \pi I(L)$, and consider the functions $\theta_{\pi\varrho}(\tau, z_0)$ contained in $R_m(K, H(K), I_0)$. We easily verify that if we write $z_0 = \pi z$, this function coincides with the function $\theta_\varrho(\tau, z)$ in $R_{m-2}(K, H(K), I(L))$, and moreover that the action of A on this function as an element of $R_m(K, H(K), I_0)$ coincides with the action on the function considered as an element of $R_{m-2}(K, H(K), I(L))$. Consequently we have

$$R_{m-2}(K, H(K), I(L)) \subset R_m(K, H(K), I_0) \subset R_m(K, H(K), I(L)).$$

In order to prove the second statement, we first prove the following

Lemma: *Let $\mathfrak{D}$ be the different of K/K_0. Every element of $I(K_0)$ is relative trace of a generating element of $\mathfrak{D}^{-1}$.*

The relative trace will be denoted by $S_{K|K_0}$. It is not difficult to show that K/K_0 admits an integral basis consisting of units. Indeed, let T be the inertia field of the extension K/K_0, $(K:T)=e_0$, $(T:K)=f_0$. For T/K_0 we take as a basis a set of elements $\omega_1, \ldots, \omega_{f_0}$ of $I(T)$ whose residue classes mod $\pi(K_0)$ constitute a basis of $I(T)/(\pi(K_0))$ over $I(K_0)/(\pi(K_0))$. These are clearly units. For K/T we take as a basis the elements $\alpha_1=1, \alpha_i=1+\pi(K)^{i-1}$, $i=2, \ldots, e_0$. Consequently the set $\beta_1, \ldots, \beta_r$ of all products $\omega_i\alpha_j$ is an integral basis for K/K_0 consisting of units. Since K_0 is properly contained in K, we have $r>1$. Now let $\beta_1^*, \ldots, \beta_r^*$ be the complementary basis, which is an $I(K_0)$-basis for $\mathfrak{D}^{-1}$. Among these β_i^* choose one, say β_1^*, whose order in $\pi(K)$ is minimal. Then clearly $\mathfrak{D}^{-1}=(\beta_1^*)$, and since β_1, β_2 are units: $\mathfrak{D}^{-1}=(\beta_1\beta_1^*)=(\beta_2\beta_1^*)$. Writing $\zeta=\beta_1\beta_1^*$, $\eta=\beta_2\beta_1^*$, we have $S_{K|K_0}(\zeta)=1$, $S_{K|K_0}(\eta)=0$. Now let ξ be an arbitrary element of $I(K_0)$. If ξ is a unit we have $\mathfrak{D}^{-1}=(\xi\zeta)$, $S_{K|K_0}(\xi\zeta)=\xi$. If ξ is not a unit, we have $\mathfrak{D}^{-1}=(\xi\zeta+\eta)$, $S_{K|K_0}(\xi\zeta+\eta)=\xi$, which proves the lemma.

Returning now to the second part of theorem 3, we take $H_0=H(K_0)$, $I_0=I(L_0)$, a generating element Δ_1 of $\mathfrak{D}$ such that $S_{K|K_0}(\Delta_1{}^{-1})=\pi^h$, and any generating element $\Delta_0 \in I(K_0)$ of the absolute different of K_0. We may suppose that $\Delta=\Delta_0\Delta_1$. Now if $\varrho \in I(K_0)$, we have

$$\theta_\varrho(\tau_0, z_0) = E_{K_0}\left(\frac{N(\varrho)\tau_0+S(\varrho z_0')}{\pi^m\Delta_0} S_{K|K_0}(\Delta_1{}^{-1})\right) = E_{K_0}\left(\frac{N(\varrho)\tau_0+S(\varrho z_0')}{\pi^{m-h}\Delta_0}\right).$$

This function can be considered as an element of $R_{m-h}(K_0, H(K_0), I(L_0)$. Again we verify that the action of A on the function, considered as an element of this space coincides with the action on the function as an element of $R_m(K, H(K_0), I(L_0))$, hence

$$R_{m-h}(K_0, H(K_0), I(L_0)) \subset R_m(K, H(K_0), I(L_0)) \subset R_m(K, H(K), I(L)).$$

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